Problems for the Middle School

Problem Editor: ATHMARAMAN R

Problem Corner

PROBLEMS FOR SOLUTION

In this edition of the problem section, all the problems are based on the fundamental concept of **divisibility** among numbers. The problems require for their solution only a basic understanding of this notion, and a basic knowledge of the rules of divisibility.

Problem V-2-M.1
What is the largest prime divisor of every three-digit number with three identical non-zero digits?

Problem V-2-M.2
Given any four distinct integers a, b, c, d, show that the product

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12.

Problem V-2-M.3
Let n be a natural number, and let $d(n)$ denote the sum of the digits of n. Show that if $d(n) = d(3n)$, then 9 divides n. Show that the converse statement is false.

Problem V-2-M.4
Let n be an arbitrary positive integer. Show that:

(a) $n^5 - n$ is divisible by 5;

(b) $n^7 - n$ is divisible by 7;

(c) $n^9 - n$ is not necessarily divisible by 9.

Problem V-2-M.5
Find all positive integers $n > 3$ such that $n^3 - 3$ is divisible by $n - 3$.

Problem V-2-M.6
Show that there cannot exist three positive integers a, b, c, all greater than 1, such that the following three conditions are simultaneously satisfied:

(a) $a^2 - 1$ is divisible by b and c;

(b) $b^2 - 1$ is divisible by c and a;

(c) $c^2 - 1$ is divisible by a and b.

Problem V-2-M.7
Using the nine nonzero digits $1, 2, 3, 4, 5, 6, 7, 8, 9$, form a nine-digit number in which each digit occurs exactly once, such that when the digits are removed one at a time starting from the units end (i.e., from the "right side"), the resulting numbers are divisible respectively by $8, 7, 6, 5, 4, 3, 2, 1$. (So if the nine-digit number is $\overline{\mathsf{ABCDEFGHI}}$, then we must have:

$$8 \mid \overline{\mathsf{ABCDEFGH}}; \qquad 7 \mid \overline{\mathsf{ABCDEFG}}; \qquad 6 \mid \overline{\mathsf{ABCDEF}}; \qquad 5 \mid \overline{\mathsf{ABCDE}};$$

and so on. Here the notation $a \mid b$ means: "a divides b".) Is the answer unique?

SOLUTIONS OF PROBLEMS IN ISSUE-V-1 (MARCH 2016)

Solution to problem V-1-M.1
Find two non-zero numbers such that their sum, their product and the difference of their squares are all equal.

Let a, b be the numbers ($a \neq 0$, $b \neq 0$). Then we have: $a + b = ab = a^2 - b^2 = (a + b)(a - b)$. Since $ab \neq 0$, $a + b \neq 0$, hence $a - b = 1$, therefore $b = a - 1$. This leads to the equation $a(a - 1) = 2a - 1$, or $a^2 - 3a + 1 = 0$. Solving this equation, we get two solution pairs:

$$a = \frac{3 + \sqrt{5}}{2}, \quad b = \frac{1 + \sqrt{5}}{2}; \qquad a = \frac{3 - \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2}.$$

Solution to problem V-1-M.2
Prove that a six-digit number formed by placing two consecutive three-digit positive integers one after the other is not divisible by any of the following numbers: 7, 11, 13. (Adapted from the Mid-Michigan Olympiad in 2014 grades 7-9)

The key observation to be made here is that if $\overline{\mathsf{ABC}}$ is a three-digit number, then the six-digit number $\overline{\mathsf{ABCABC}}$ is necessarily a multiple of 1001. This should be clear since

$$\overline{\mathsf{ABCABC}} = \overline{\mathsf{ABC}000} + \overline{\mathsf{ABC}} = \overline{\mathsf{ABC}} \times 1000 + \overline{\mathsf{ABC}} = \overline{\mathsf{ABC}} \times 1001.$$

Since $1001 = 7 \times 11 \times 13$, the above equality implies that $\overline{\mathsf{ABCABC}}$ is divisible by each of the numbers $7, 11, 13$. This further implies that if $\overline{\mathsf{DEF}} = \overline{\mathsf{ABC}} + 1$, then the six-digit number $\overline{\mathsf{ABCDEF}}$ will leave remainder 1 when divided by any of the numbers $7, 11, 13$. This means in particular that $\overline{\mathsf{ABCDEF}}$ will not be divisible by any of the numbers $7, 11, 13$.

Solution to problem V-1-M.3

If n is a whole number, show that the last digit in $3^{2n+1} + 2^{2n+1}$ *is* 5.

Note that $3^{2n+1} + 2^{2n+1}$ is an odd number. Also note the general fact that if k is odd, then $a^k + b^k$ is divisible by $a + b$. Hence $3^{2n+1} + 2^{2n+1}$ is divisible by 5. It follows that the last digit in $3^{2n+1} + 2^{2n+1}$ is 5.

Solution to problem V-1-M.4

(a) *Show that the sum of any m consecutive squares cannot be a square for* $m \in \{3, 4, 5, 6\}$.

Let S_m denote the sum of m consecutive squares:

$$S_m = n^2 + (n+1)^2 + \cdots + (n+m-1)^2.$$

We now consider each of the cases in turn.

- $S_3 = n^2 + (n+1)^2 + (n+2)^2 = 3n^2 + 6n + 5$, which is of the form $3k + 2$. However, no square is of this form. Hence S_3 cannot be a perfect square.
- In the same way, we get $S_4 = 4n^2 + 12n + 14$, which is of the form $4k + 2$. However, no square is of this form. Hence S_4 cannot be a perfect square.
- In the same way, we get $S_5 = 5n^2 + 20n + 30 = 5\left(n^2 + 4n + 6\right)$, which is a multiple of 5. If S_5 is a square, then it must be a multiple of 25, hence $n^2 + 4n + 6$ must be a multiple of 5. But $n^2 + 4n + 6 = (n+2)^2 + 2$, so for $n^2 + 4n + 6$ to be a multiple of 5, $(n+2)^2$ must be of the form $5k + 3$. However, no square is of this form. Hence S_5 cannot be a perfect square.
- In the same way, we get $S_6 = 6n^2 + 30n + 55 = 6n(n+5) + 55$, which is of the form $12k + 7$. However, no square is of this form. Hence S_6 cannot be a perfect square.

Observe that in each of the above cases, we followed a similar strategy. But in each case, the precise path taken was slightly different.

(b) *Can the sum of* 11 *consecutive square numbers be a square number?*

In the same way, we get $S_{11} = 11n^2 + 110n + 385 = 11\left(n^2 + 10n + 35\right)$. For this to be a square, $n^2 + 10n + 35$ must have the form $11m^2$. Since $n^2 + 10n + 35 = (n+5)^2 + 10$, this demand leads to the equation $x^2 - 11m^2 = -10$. This equation does indeed have solutions! For example, $x = 1$, $m = 1$ is a solution; but it yields $n = -4$, a negative value. The next solution after this one is $x = 23$, $m = 7$, which yields $n = 18$, $m = 7$. So: $18^2 + 19^2 + 20^2 + 21^2 + 22^2 + 23^2 + 24^2 + 25^2 + 26^2 + 27^2 + 28^2 = 5929 = 77^2$.

Solution to problem V-1-M.5

(a) *Which positive integers have exactly two positive divisors? Which have three positive divisors?*

The positive integers which have exactly two divisors are clearly the prime numbers. (The prime number p has two divisors 1 and p itself.)

Now we consider the case when the positive integer n has exactly three divisors. From the above, we know that n is not a prime number. Suppose that n can be written in the form $n = ab$ where $1 < a < b < n$. In this case, n has at least the following four divisors: $1, a, b, n$. So if n is to have exactly three divisors, then it must be the case that n has a divisor a with $1 < a < n$; but it should also be the case that n cannot be written in the form $n = ab$ where $1 < a < b < n$.

The only option allowed by these restrictions is that n must be the square of a prime number. Indeed, if $n = p^2$ where p is a prime number, then n has just the following three divisors: $1, p, p^2$.

(b) *Among integers a, b, c, each exceeding 20, one has an odd number of divisors, and each of the other two has three divisors. If $a + b = c$, find the least value of c.*

The integers having an odd number of divisors are the perfect squares. Drawing from the result in part (a), we may suppose that the three numbers are n^2, p^2, q^2 where p, q are prime numbers. We are told that the sum of two of these squares equals the third square. The equation $p^2 + q^2 = n^2$ does not yield any solution; for if p, q are both odd, then we get $n^2 \equiv 2 \pmod 4$, which is not possible; and if $p = 2$, then we get $n^2 - q^2 = 4$; but this yields no solutions. So we consider instead the equation $p^2 + n^2 = q^2$. This turns out to have numerous solutions:

$$(p, n, q) = (3, 4, 5),\ (5, 12, 13),\ (11, 60, 61),\ (19, 180, 181),\ \ldots,$$

i.e.,

$$(a, b, c) = (9, 16, 25),\ (25, 144, 169),\ (121, 3600, 3721),\ \ldots$$

The first of these possibilities is ruled out, as it is given that a, b, c all exceed 20; so we must choose the second triple, which means that the least value of c is 169.

Solution to problem V-1-M.6

A group of 43 devotees consisting of ladies, men and children, went to a temple. After a ritual, the priest distributed 229 flowers to the visitors. Each lady got 10 flowers, each man got 5 flowers and each child got 2 flowers. If the number of men exceeded 10 but not 15, find the number of women, men and children in the group.

Let there be a ladies, b men and c children; then $a + b + c = 43$ and $10a + 5b + 2c = 229$; also $10 < b \le 15$. The equations may be rewritten as: $a + b = 43 - c$, $10a + 5b = 229 - 2c$. Treating these as a pair of simultaneous equations in a, b and solving them in the usual manner, we get:

$$a = \frac{3c + 14}{5}, \quad b = \frac{201 - 8c}{5}.$$

Since $10 < b \le 15$, we get

$$10 < \frac{201 - 8c}{5} \le 15, \quad \therefore \quad \frac{63}{4} \le c < \frac{151}{8},$$

i.e., $16 \le c \le 18$, since c is an integer. Also, since a is an integer, we have:

$$3c \equiv -14 \pmod 5 \equiv 6 \pmod 5, \qquad \therefore \quad c \equiv 2 \pmod 5.$$

The two conditions yield: $c = 17$, and therefore $a = 13$, $b = 13$. So there are 13 ladies, 13 men and 17 children.

Solution to problem V-1-M.7

There are two towns, A and B. Person P travels from A to B, covering half the distance at rate a, and the remaining half at rate b. Person Q travels from A to B (starting at the same time as P), traveling for half the time at rate a, and for half the time at rate b. Who reaches B earlier?

Let d be the distance between the two towns. The time T_P taken by P is given by

$$T_P = \frac{d/2}{a} + \frac{d/2}{b} = \frac{d}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{d(a+b)}{2ab} = \frac{d}{\text{harmonic mean of } a, b}.$$

Let T_Q be the time taken by Q; then we have:

$$d = \frac{aT_Q}{2} + \frac{bT_Q}{2} = \frac{T_Q}{2}(a+b), \therefore \quad T_Q = \frac{2d}{a+b} = \frac{d}{\text{arithmetic mean of } a, b}.$$

For arbitrary positive numbers a, b, it is true that the arithmetic mean of a, b is greater than or equal to the harmonic mean of a, b; hence $T_Q \le T_P$.